Indag. Mathem., N.S., 2 (3), 275-281 September 30, 1991

On the realization of forms of affine Kac-Moody algebras

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Communicated by Prof. T.A. Springer at the meeting of March 25, 1991

1. INTRODUCTION

The main reference we shall use is [K]. Let k be a field of char 0, $\otimes\bar{k}$ an algebraic closure. Let $A=(a_{ij})\in\mathbb{Z}^{n\times n}$ be a generalized Cartan matrix of affine type [K, 1.1, 4.3]. Let $g_k(A)$ denote the Kac-Moody algebra associated to A [K, 1.3] and $g_k'(A)=[g_k(A),g_k(A)]$. Concrete realizations of such Kac-Moody algebras are known; cf. [K, chapters 7, 8] and 2 below. For example, if A is "non-twisted", $g_k'(A)$ is a central extension of the loop algebra of the corresponding finite dimensional Lie algebra.

In this paper, we shall present a construction of certain forms of affine Kac-Moody algebras; i.e. Lie algebras over k which after extension of scalars to $\bar{k}$ become isomorphic to $g_{\bar{k}}'(A)$ or $g_{\bar{k}}(A)$. The point of the construction is to consider algebras of the form $k[x,y]/\langle px^2+qy^2-1\rangle$ instead of the loop algebra $k[t,t^{-1}]$; details are presented in 3. This construction seems to provide new examples even in the real case, see 4. In 5, we relate the introduced forms to the presentation by generators and relations in [A], in the simplest case $A=A_1^{(1)}$.

I thank G. Rousseau for interesting conversations and A. Guichardet for questions about [A] which are in the origin of 5.

2. REALIZATION OF AFFINE KAC-MOODY ALGEBRAS

Let $\mathscr{A}$ be an associative commutative k-algebra, a a finite dimensional Lie

This work was partially supported by CONICET and FAMAF (Argentina).

algebra over k. We shall consider $a \otimes_k \mathscr{A}$ as a Lie algebra over k, with the bracket

$$[x \otimes u, y \otimes v] = [x, y] \otimes uv.$$

In particular, we shall denote as usual $L_k(a) = a \otimes k[t, t^{-1}]$ (We will drop the subscript k when it is not necessary).

More generally, let $\theta = (a_h)_{h \in \mathbb{Z}_m}$ be a grading of a; i.e. $a = \oplus a_h$, $[a_h, a_i] \subseteq a_{h+i}$. Then $L(a, \theta)$ is the Lie subalgebra of $L(a)$ defined by $L(a, \theta) = \oplus_{h \in \mathbb{Z}} a_{\bar{h}} \otimes kt^h$, where $\bar{h}$ is the class of h modulo m.

Now let us recall some standard facts about central extensions of Lie algebras, see for example [G] for details. Let V be a k-vector space. A central extension of a by V is an exact sequence of Lie algebras

$$0 \to V \to e \to a \to 0$$

such that V is contained in the center of e. It is well-known that (suitably defined) equivalence classes of extensions of a by V are parametrized by $H^2(a, V)$, the second cohomology group of a with respect to V, regarded as trivial a-module.

In general, let $(\,|\,)$ be a symmetric invariant bilinear form on a and $\{,\} : \mathscr{A} \times \mathscr{A} \to k$ a bilinear skew symmetric form satisfying

$$\text{(C)} \qquad \{PQ, R\} + \{PR, Q\} + \{QR, P\} = 0 \quad \forall P, Q, R \in \mathscr{A}.$$

Then $\tau : a \otimes \mathscr{A} \times a \otimes \mathscr{A} \to k$, given by

$$\tau(x \otimes P, y \otimes Q) = (x \mid y)\{P, Q\}$$

is a 2-cocycle, i.e. τ defines a central extension $a \otimes \mathscr{A} \oplus kc^\tau$ of $a \otimes \mathscr{A}$ by

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ + \tau(x \otimes P, y \otimes Q)c^\tau.$$

In addition, let $\mathscr{D}_0$ be a derivation of $\mathscr{A}$ satisfying

$$\text{(D)} \qquad \{\mathscr{D}_0 P, Q\} + \{P, \mathscr{D}_0 Q\} = 0 \quad \forall P, Q \in \mathscr{A}.$$

It is known (cf. [K, Prop. 7.3]) that this is equivalent (if $(\,|\,)$ is non-trivial) to the fact that $\mathscr{D} : a \otimes \mathscr{A} \oplus kc^\tau \to a \otimes \mathscr{A} \oplus kc^\tau$ given by

$$\mathscr{D}\,\big|_{a \otimes \mathscr{A}} = id \otimes \mathscr{D}_0 \quad \mathscr{D}(c^\tau) = 0$$

is a derivation of $a \otimes \mathscr{A} \oplus kc^\tau$. Therefore, we can define a Lie algebra structure on $a \otimes \mathscr{A} \oplus kc^\tau \oplus k\mathscr{D}$ by imposing $a \otimes \mathscr{A} \oplus kc^\tau$ to be a subalgebra and

$$[\mathscr{D}, x] = \mathscr{D}(x) \quad \forall x \in a \otimes \mathscr{A} \oplus kc^\tau.$$

Now let us consider the particular case when $a = g$ is a simple finite dimensional split Lie algebra, whose Dynkin diagram will be denoted X_n, and $\mathscr{A} = k[t, t^{-1}]$. It is well-known that $L(g)$ admits a central extension by k, i.e. that $H^2(L(g), k)$ is non-trivial. Indeed, let $(\,|\,)$ be the Killing form (or a multiple of it), and $\{P, Q\} = Res(PdQ)$:

$$\{t^n, t^m\} = m\delta_{m+n,0}.$$

It is easy to see that $\{,\}$ satisfies (C). We shall denote $\tilde{L}(g)$ the central extension of $L(g)$ corresponding to this choice. We will write c instead of c^{τ}. More generally, $\tilde{L}(g,\theta)$ will denote the subalgebra $L(g,\theta)\oplus kc$ of $\tilde{L}(g)$. It is also known that $\dim H^2(g\otimes k[t,t^{-1}],k)=1$, see [K, Exercise 7.7] or [G].

Now let D_0 be the derivation of $k(t,t^{-1}]$ defined by $D_0(t^h)=ht^{(h)}: D_0$ satisfies (D). Thus we are able to consider the Lie algebra $\hat{L}(g)=\tilde{L}(g)\oplus kD$ and its subalgebra $\hat{L}(g,\theta)=\tilde{L}(g,\theta)\oplus kD$.

THEOREM. (Kac-Moody; *cf.* [K, *ch.* 7, 8]). *Let θ correspond to a diagram automorphism of order h. Then $\hat{L}(g,\theta)$ is isomorphic to the Kac-Moody algebra corresponding to the Dynkin diagram $X_n^{(h)}$.*

3. THE CONSTRUCTION

From now on, let g denote an absolutely simple finite dimensional Lie algebra over k, $\bar{g}=g\otimes_k\bar{k}$. Let X_n be the Dynkin diagram corresponding to $\bar{g}$. Let $\mathscr{A}=k[u,v]/\langle pu^2+qv^2-1\rangle$ denote the coordinate ring of the conic defined by the equation $pu^2+qv^2=1$, $p,q\in k-0$. Clearly, $L(a)$ is isomorphic to $a\otimes\mathscr{A}$ if $\mathscr{A}$ corresponds to the conic $u^2-v^2=1$. Let us remark that $(g\otimes\mathscr{A})\otimes\bar{k}$ is (non-canonically) isomorphic to $L_{\bar{k}}(\bar{g})$. Even more, $(g\otimes\mathscr{A})\otimes k'$ is isomorphic to $L_{k'}(g\otimes k')$, if $k'=k(\sqrt{-pq})$ and $\sqrt{-pq}\in\bar{k}$ is a fixed square root of $-pq$. For this, let us fix the isomorphism $\phi:\mathscr{A}\otimes k'\to k'[t,t^{-1}]$ defined by $\phi^{-1}(t)=pu+\sqrt{-pq}v$, $\phi^{-1}(t^{-1})=u-\sqrt{-pq}/pv$. Thanks to ϕ, we have

$$(g\otimes\mathscr{A})\otimes k'\simeq g\otimes_k k'[t,t^{-1}]\simeq(g\otimes_k k')\otimes_{k'}k'[t,t^{-1}]\simeq L_{k'}(g\otimes k').$$

The following fact is standard:

LEMMA. Let k' be a finite field extension of k. Then

$$H^2(a,V)\otimes k'\simeq H^2(a\otimes k', V\otimes k').$$

PROOF. Let U (resp., U') be the universal enveloping algebra of a (resp., of $a\otimes k'$). Let

$$\cdots\to P_2\to P_1\to P_0\to V\to 0$$

be a projective resolution of V as U-module. Then

$$\cdots\to P_2\otimes k'\to P_1\otimes k'\to P_0\otimes k'\to V\otimes k'\to 0$$

is a projective resolution of $V\otimes k'$ as U'-module. There is a natural isomorphism

$$Hom_U(a,V)\otimes k'\to Hom_{U'}(a\otimes k', V\otimes k')$$

given by

$$T\otimes\lambda\mapsto(v\otimes\mu\mapsto T(v)\otimes\lambda\mu).$$

(Let $S \in Hom_{U'}(a \otimes k', V \otimes k')$, let $\{e_i\}$ be a k-basis of k'. Then $S(v \otimes 1) = \sum S_i(v) \otimes e_i$ and S is the image of $\sum S_i \otimes e_i$. On the other hand, if the image of $\sum T_j \otimes f_j$ is zero, we rewrite $\sum T_j \otimes f_j = \sum S_i \otimes e_i$; hence $\sum S_i(v) \otimes e_i = 0$, and $S_i(v) = 0$, $\forall i, v$. Therefore $\sum T_j \otimes f_j = 0$). From this, the Lemma follows.

We shall apply the preceding analysis to $g \otimes \mathscr{A}$ instead of a and $k' = k(\sqrt{-pq})$. It follows from the Lemma that there exists a non-zero cocycle $\tau_{p,q} \in H^2(g \otimes \mathscr{A})$ which via the non-canonical isomorphism of the Lemma, is a multiple of τ, say, $\tau_{p,q} = \varepsilon\tau$. More precisely, if $(|)_0$ is a bilinear invariant form on g such that $(|)$ restricted to g is $(|)_0$ (for example, if both $(|)_0$ and $(|)$ are the respective Killing forms) and $\{,\}_0 \colon \mathscr{A} \times \mathscr{A} \to k'$ is the restriction of $\varepsilon^{-1}\{,\}$, then $\{\mathscr{A}, \mathscr{A}\}_0 \subset k$ and $\{,\}_0$ is a k-bilinear skew symmetric form satisfying (C).

Thus $g \otimes \mathscr{A} \oplus kc^{\tau_{p,q}}$, the central extension of $g \otimes \mathscr{A}$ constructed via the cocycle $\tau_{p,q}$ is a k-form of $\tilde{L}_{\bar{k}}(\bar{g})$. On the other hand, let us consider the unique derivation $D^{p,q}$ of $k[u, v]$ such that

$$D^{p,q}(u) = -qv \quad D^{p,q}(v) = pu.$$

Then $D^{p,q}(pu^2 + qv^2 - 1) = 0$ and hence $D^{p,q}$ gives raise to a derivation $d_0^{p,q}$ of $\mathscr{A} = k[u, v]/\langle pu^2 + qv^2 - 1\rangle$. Let us recall that we have identified $\mathscr{A} \otimes k'$ with $k'[t, t^{-1}]$ via ϕ and hence

$$t = pu + \sqrt{-pq}v \Rightarrow (d_0^{p,q} \otimes id)(t) = \sqrt{-pq}t$$

$$t^{-1} = u - \frac{\sqrt{-pq}}{p}v \Rightarrow (d_0^{p,q} \otimes id)(t^{-1}) = -\sqrt{-pq}t^{-1}.$$

In other words, $d_0^{p,q} \otimes id$ is $\sqrt{-pq}D_0$, where D_0 is as in 2. In particular, we obtain that $d_0^{p,q}$ satisfies (D) and hence we have a derivation $d^{p,q}$ of $g \otimes \mathscr{A} \oplus kc^{\tau_{p,q}}$. As in 2, we obtain a Lie algebra structure on $g \otimes \mathscr{A} \oplus kc^{\tau_{p,q}} \oplus kd^{p,q}$ by imposing $g \otimes \mathscr{A} \oplus kc$ to be a subalgebra and

$$[d^{p,q}, x] = d^{p,q}(x) \quad \forall x \in g \otimes \mathscr{A} \oplus kc^{\tau_{p,q}}.$$

Clearly $g \otimes \mathscr{A} \oplus kc^{\tau_{p,q}} \oplus kd^{p,q}$ is a form of the Kac-Moody algebra corresponding to $X_n^{(1)}$.

On the other hand, the differential operator $(d^{p,q})^2$ is diagonalisable in $\mathscr{A}$, with eigenvalues $-pqn^2$, $n \in \mathbb{Z}$; let $\mathscr{A}_n$ be the corresponding eigenspace. Then $\mathscr{A}_n \mathscr{A}_m \subseteq \mathscr{A}_{n+m} \oplus \mathscr{A}_{n-m}$. Let $\theta = (g_h)_{h \in \mathbb{Z}_2}$ be a $\mathbb{Z}_2$-grading of g. It follows that $\oplus_{h \in \mathbb{Z}} g_{\bar{h}} \otimes \mathscr{A}_h$ (resp., $\oplus_{h \in \mathbb{Z}} g_{\bar{h}} \oplus kc^{\tau_{p,q}}$, $\oplus_{h \in \mathbb{Z}} g_{\bar{h}} \otimes \mathscr{A}_h \oplus kc^{\tau_{p,q}} \oplus kd^{p,q}$) is a form of $L(\bar{g}, \theta)$ (resp., $\tilde{L}(\otimes \bar{g}, \theta)$, $\hat{L}(\otimes \mathscr{A}_h, \theta)$).

4. THE REAL CASE

If $k = \mathbb{R}$, there are only three different isomorphism classes of such $\mathscr{A}$, namely, $\mathbb{R}[t, t^{-1}]$; $\mathbb{R}[u, v]/\langle u^2 + v^2 - 1\rangle = \mathbb{R}[S^1]$ and $\mathbb{R}[u, v]/\langle -u^2 - v^2 - 1\rangle$. In the second case, we can choose (cf. [KP])

$$\{,\}_0 \colon \mathbb{R}[S^1] \times \mathbb{R}[S^1] \to \mathbb{R}$$

$$\{P, Q\}_0 = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d}{dt} P(e^{it})\right) Q(e^{it}) dt.$$

The forms corresponding to the first two cases are known (see [GW], [R3], [R4]); the third seems to be not considered yet, perhaps because $\mathscr{A}$ has no real points. Another way to obtain such forms is to consider the antilinear involution of $L_{\mathbb{C}}(\bar{g})$ given by

$$x \otimes t^j \mapsto (-1)^j \gamma(x) \otimes t^{-j} \quad x \in \bar{g}, j \in \mathbb{Z},$$

where γ is the antilinear involution of $\bar{g}$ whose fixed point set is g. (Compare with [R3, 1.4, 2.6]).

REMARK. G. Rousseau pointed out to me that the third form is implicitly considered in [R2], where a correspondance between real forms and linear involutions is established; the corresponding linear involution is given in [BR].

5. GENERATORS AND RELATIONS

From now on we shall restrict our attention to the following generalized Cartan matrix:

$$A = (a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

It corresponds to the extended Dynkin diagram of A_1 and is denoted $A_1^{(1)}$.

On the other hand, in [A] a presentation was given by generators and relations of some forms of Kac-Moody algebras. In the present case, it runs as follows: let us fix elements a_1, a_2, b_1, b_2, s of k^* satisfying

$$\frac{b_2}{a_2} = \frac{b_1}{a_1} s^2.$$

Let $g(a_i, b_i, s)$ denote the Lie algebra over k generated by $\{X_i, Y_i, Z_i : i = 1, 2\}$ with relations

$$\begin{array}{ll} [Z_1, Z_2] = 0 & [X_i, Y_i] = 2Z_i \\ [Y_i, Z_i] = -2b_i X_i & [Z_i, X_i] = -2a_i Y_i \\ [Y_2, Z_1] = 2b_1 s X_2 & [Z_1, X_2] = 2a_1 s^{-1} Y_2 \\ [Y_1, Z_2] = 2b_2 s^{-1} X_1 & [Z_2, X_1] = 2a_2 s Y_1 \\ [X_1, Y_2] = s[Y_1, X_2] & [X_1, X_2] = -a_1 b_1^{-1} s^{-1} [Y_1, Y_2] \end{array}$$

and if $i \neq j$:

$$(adX_i)^3 X_j = 4a_i (adX_i) X_j$$

$$(adX_i)^3 Y_j = 4a_i (adX_i) Y_j.$$

(This presentation is slightly more precise than in [A]; see [A2]).

We shall relate now the two approaches. For this, we shall need to consider the following basis of $\mathscr{A} = k[u, v]/\langle pu^2 + qv^2 - 1\rangle$. We recall that $\mathscr{A} \otimes k' \simeq k'[t, t^{-1}]$ via

$$t \mapsto pu + \sqrt{-pq}\,v \quad t^{-1} \mapsto u - \frac{\sqrt{-pq}}{p} v.$$

Let j be a positive integer. We define $s_j,\ c_j \in \mathscr{A}$ by

$$t^j = c_j + \sqrt{-pq}\, s_j .$$

We have

$$c_1 = pu \quad s_1 = v \quad c_{i+j} = c_i c_j - pq s_i s_j \quad s_{i+j} = c_i s_j + c_j s_i .$$

It follows easily by induction that $t^j (c_j - \sqrt{-pq}\, s_j) = c_j^2 + pq s_j^2 = p^j \in k - 0$; thus $t^{-j} = p^{-j}(c_j - \sqrt{-pq}\, s_j)$.

LEMMA. *The family* $(1, c_j, s_j)$ *is a basis of* $\mathscr{A}$. *Moreover, if we require the cocycle* $\tau^{p,q}$ *to be* $1/\sqrt{-pq}\tau$, *then*

$$\{c_j, c_i\}_0 = 0 \quad \{s_j, s_i\}_0 = 0 \quad \{c_j, s_i\}_0 = \frac{jp^j}{2pq}\delta_{i,j} .$$

PROOF. It can be deduced easily from the following equalities:

$$c_j = \frac{t^j + p^j t^{-j}}{2}, \quad s_j = \frac{t^j - p^j t^{-j}}{2\sqrt{-pq}} .$$

Let $sq(a_1, b_1)$ be a Lie algebra over k having a basis x, y, z and a multiplication table

$$[x, y] = 2z \quad [y, z] = -2b_1 x \quad [z, x] = -2a_1 y .$$

It is known that $sq(a_1, b_1)$ is a form of $sl(2)$. Let us fix the bilinear invariant form on $sq(a_1, b_1)$ given by

$$(x|x) = 2a_1, \quad (y|y) = 2b_1, \quad (z|z) = -2a_1 b_1,$$

$$(x|y) = (x|y) = (y|z) = 0.$$

PROPOSITION. $g(a_i, b_i, s)$ *is isomorphic to* $sq(a_1, b_1) \otimes \mathscr{A} \oplus kc$, *where*

$$\mathscr{A} = k[u, v] \Big/ \left\langle a_1 a_2 u^2 + \frac{b_1}{a_2} v^2 - 1 \right\rangle ,$$

with the bracket

$$[r \otimes P,\ s \otimes Q] = [r, s] \otimes PQ + (r|s)\{P, Q\}_0 c,$$

where $\{,\}_0$ *is as above and is normalized, as in the Lemma, by* $\{u, v\}_0 = 1/2a_1 b_1$.

PROOF. Let us consider the application from the free Lie algebra in variables $\{X_i, Y_i, Z_i : i = 1, 2\}$ into $sq(a_1, b_1) \otimes \mathscr{A} \oplus kc$ given by

$$X_1 \mapsto x \otimes 1 \quad X_2 \mapsto x \otimes a_2 u + y \otimes v$$

$$Y_1 \mapsto y \otimes 1 \quad Y_2 \mapsto s\left(\frac{b_1}{a_1} x \otimes v - y \otimes a_2 u\right)$$

$$Z_1 \mapsto z \otimes 1 \quad Z_2 \mapsto \frac{s a_2}{a_1}(-z \otimes 1 + c).$$

We need to check the above relations. This a tedious but straightforward task, which will be omitted. Thus we have a map $g(a_i, b_i, s) \to sq(a_1, b_1) \otimes \mathscr{A} \oplus kc$; it is possible to see that it is an isomorphism.

REMARK. If $a_1 = a_2 = a$, $b_1 = b_2 = b$, $s = 1$, then $g(a_i, b_i, s) = g(a, b)$ is isomorphic to $sq(a, b) \otimes k[u, v]/\langle a^2u^2 + (b/a)v^2 - 1\rangle \oplus kc$. In particular, we can give a negative answer to the following question of Alain Guichardet: does $sq(a, b) \simeq sq(c, d)$ imply that $g(a, b) \simeq g(c, d)$? Indeed, $sq_{\mathbb{R}}(1, -1) \simeq sq_{\mathbb{R}}(1, 1) \simeq sl(2, \mathbb{R})$ but $g(1, -1) \otimes \mathbb{R}[, t, t^{-1}] \oplus \mathbb{R}c$ is a form of first kind and $g(1, 1) \otimes \mathbb{R}[\mathbb{S}^1] \oplus \mathbb{R}ic$ is a form of second kind (cf. [R3]).

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